Pergamon

PII: S0021-8928(98)00006-9

J. Appl. Maths Mechs, Vol. 62, No. 1, pp. 51–65, 1998

0021-8928/98 $24.00+0.00

CONTINUATION OF PERIODIC MOTIONS OF A REVERSIBLE SYSTEM IN NON-STRUCTURALLY STABLE CASES. APPLICATION TO THE *N*-PLANET PROBLEM†

V. N. TKHAI

Moscow

(*Received* 23 *December* 1996)

The problem of continuing symmetric periodic solutions of an autonomous or periodic reversible system with respect to a parameter is solved. Non-structurally stable cases, when the generating system does not guarantee that the solution can be continued, are considered. Three approaches are proposed to solving the problem: (a) particular consideration of terms that depend on the small parameter and the selection of generating solutions; (b) the selection of a generating system depending on the small parameter; (c) reduction to a quasi-linear system which is then analysed using the first approach. Within the framework of the third approach the existence of a periodic motion is also established that differs from the generating one by a quantity whose order is a fractional power of the small parameter. The theoretical results are used to prove the existence of two families of periodic three-dimensional orbits in the *N*-planet problem. The orbit of each planet is nearly elliptical and situated in the neighbourhood of its fixed plane; the angle between the planes is arbitrary. The average motions of the planets in these orbits relate to one another as natural numbers (the resonance property), and at instants of time that are multiples of the half-period the planets are either aligned in a straight line—the line of nodes (the first family), or cross the same fixed plane (the second family). The phenomenon of a parade of planets is observed. The planets' directions of motion in their orbits are independent.

1. PERIODIC MOTIONS OF A REVERSIBLE SYSTEM

We consider an autonomous or 2π-periodic reversible system

$$\mathbf{u}^{\cdot} = \mathbf{U}_0(\mathbf{u}, \mathbf{v}, t) + \mu \mathbf{U}_1(\mu, \mathbf{u}, \mathbf{v}, t) \tag{1.1}$$

$$\mathbf{v}^{\cdot} = \mathbf{V}_0(\mathbf{u}, \mathbf{v}, t) + \mu \mathbf{V}_1(\mu, \mathbf{u}, \mathbf{v}, t); \quad \mathbf{u} \in \mathbb{R}^l, \quad \mathbf{v} \in \mathbb{R}^n \quad (l \geqslant n)$$

with fixed set $\mathbf{M} = \{\mathbf{u}, \mathbf{v} : \mathbf{v} = \mathbf{0}\}$ [11] and small parameter μ. Let us assume that the generating system, obtained from (1.1) by putting $\mu = 0$, has a $2T^*$-periodic solution ($T^* = \pi$ for a 2π-periodic system) symmetric relative to $\mathbf{M}$

$$\mathbf{u} = \boldsymbol{\varphi}(t), \quad \mathbf{v} = \boldsymbol{\psi}(t); \quad \boldsymbol{\varphi}(-t) = \boldsymbol{\varphi}(t), \quad \boldsymbol{\psi}(-t) = -\boldsymbol{\psi}(t) \tag{1.2}$$

We seek the conditions under which, for sufficiently small $\mu \neq 0$, system (1.1) will also have a solution, symmetric relative to $\mathbf{M}$, that is identical with (1.2) at $\mu = 0$. In that case we shall refer to (1.2) as a generating solution.

Let $\mathbf{u}(\mu, u_1^\circ, \ldots, u_l^\circ, t)$, $\mathbf{v}(\mu, u_1^\circ, \ldots, u_l^\circ)$ denote a symmetric solution of system (1.1) with initial data $u_1^\circ, \ldots, u_l^\circ$, $\mathbf{v}^\circ = \mathbf{0}$. Then a necessary and sufficient condition for this solution to be $2T$-periodic is [1]

$$v_s(\mu, u_1^\circ, \ldots, u_l^\circ, T) = 0 \quad (s = 1, \ldots, n) \tag{1.3}$$

Hence it follows that in the general case the symmetric periodic motions of a reversible system (1.1) form an $(m+1)$-family, where $m \geqslant l - n$. The parameters of this family are at least $l - n$ of the initial data $u_1^\circ, \ldots, u_l^\circ$ and μ; in the case of an autonomous system the parameter T must also be added. In the generating system we have $\mu = 0$ and the dimensionality of the family of periodic motions to which solution (1.2) belongs is m. Let us assume that $h_1, \ldots, h_m$ are the parameters of the family and that

†*Prikl. Mat. Mekh.* Vol. 62, No. 1, pp. 56–72, 1998.

solution (1.2) corresponds to parameter values $h_1^*, \ldots, h_m^*$. In an autonomous system the half-period T either depends on the parameters $h_1, \ldots, h_m$ and $T(h_1^*, \ldots, h_m^*) = T^*$, or T is added to h_j $(j = 1, \ldots, m)$ as a parameter.

Express Eq. (1.3) as

$$v_s(0,\mathbf{u}^*,T^*)+\sum_{j=1}^{l}(F_{sj}^*+F_{sj})\Delta u_j^\circ+(G_s^*+G_s)\Delta T+(Q_s^*+Q_s)\mu=0 \quad (s=1,\ldots,n) \tag{1.4}$$

$$F_{sj}^*=\left(\frac{\partial v_s}{\partial u_j^\circ}\right)_*, \quad G_s^*=\left(\frac{\partial v_s}{\partial T}\right)_*, \quad Q_s^*=\left(\frac{\partial v_s}{\partial \mu}\right)_*$$

(the asterisk indicates that the relevant quantities are evaluated at $\mu = 0$, $u_j^\circ = u_j^* = \varphi_j(0)$), where the functions F_{sj}, G_s, Q_s vanish at $\mu = 0$, $\Delta T = T - T^* = 0$, $\Delta\mathbf{u}^\circ = \mathbf{u}^\circ - \mathbf{u}^* = \mathbf{0}$.

Obviously, $v_s(0, \mathbf{u}^*, T^*) = 0$ $(s = 1, \ldots, n)$. In addition, in a periodic system one has to put $\Delta T = 0$. Put

$$\mathrm{Ra}=\mathrm{rank}\|F_{sj}^*\|, \quad \mathrm{Ra}_1=\mathrm{rank}\|F_{sj}^*,G_s^*\| \tag{1.5}$$

If $\mathrm{Ra} = n$, system (1.4) is solvable for n of the quantities $\Delta u_1^\circ, \ldots, \Delta u_l^\circ$. This means that the generating system, together with solution (1.2), has an $(l-n)$-family of $2T^*$-periodic symmetric solutions, each of which is a generating solution [2]. In an autonomous system, the condition $\mathrm{Ra} = n$ guarantees the existence, for sufficiently small μ, of an $(l-n)$-family of $2T$-periodic systems of (1.1) [3]. In that case T is independent of μ and $\mathbf{u}^\circ$ and lies in some interval containing T^*. If $\mathrm{Ra} = n-1$, $\mathrm{Ra}_1 = n$, the period of the symmetric periodic solutions of system (1.1) depends on μ (in the general case, on $\mathbf{u}^\circ$ as well), and if $\mu \neq 0$ one cannot guarantee the existence of $2T^*$-periodic solutions.

The following conclusion is important. In each of the situations considered: (a) $\mathrm{Ra} = n$, (b) $\mathrm{Ra} = n-1$, $\mathrm{Ra}_1 = n$ and the system is autonomous, contains a non-isolated case (in Poincaré's sense), but nevertheless the property of a reversible generating system to have a symmetric periodic motion is structurally stable if the perturbations are also reversible.

Now let $\mathrm{Ra} = k < n$. In that case, to fix our ideas, we solve the first k equations of (1.3) for $u_1^\circ, \ldots, u_k^\circ$ and substitute the result into the remaining equations, obtaining

$$v_s(0,u_{k+1}^\circ,\ldots,u_l^\circ,T)+\mu f_s(\mu,u_{k+1}^\circ,\ldots,u_l^\circ,T)=0 \quad (s=n-k+1,\ldots,n) \tag{1.6}$$

In the case of a 2π-periodic system (1.1) we have $T = \pi$, and when $\mu = 0$ system (1.3) has a family of solutions which depend on m arbitrary parameters. The same is true of system (1.6), and so

$$v_s(0,u_{k+1}^\circ,\ldots,u_l^\circ)\equiv 0 \quad (s=n-k+1,\ldots,n)$$

Therefore, if (1.2) is a generating solution of the generating system, its parameters $h_1^*, \ldots, h_m^*$ must satisfy the equalities

$$P_\nu(h_1^*,\ldots,h_m^*)\equiv f_\nu(0,u_{k+1}^\circ(h_1^*,\ldots,h_m^*),\ldots,u_l^\circ(h_1^*,\ldots,h_m^*),\pi)=0 \quad (\nu=n-k+1,\ldots,n) \tag{1.7}$$

Subject to the condition

$$\mathrm{Ra}^*=\mathrm{rank}\left\|\frac{\partial P_\nu(h_1^*,\ldots,h_m^*)}{\partial h_j^*}\right\|=n-k \tag{1.8}$$

system (1.6), considered for sufficiently small μ, has a solution $h_j = h_j(\mu)$, $h_j(0) = h_j^*$ $(j = 1, \ldots, m)$. Thus Eqs (1.7) determine the parameter values $h_1^*, \ldots, h_m^*$ of a generating solution if the latter satisfy condition (1.8).

In an autonomous system, we distinguish two non-structurally stable cases: (1) $\mathrm{Ra} = \mathrm{Ra}_1 = k < n$, (2) $\mathrm{Ra} = k$, $\mathrm{Ra}_1 = k+1 < n$, the first of which necessarily occurs when the half-period T of the family depends on $h_1, \ldots, h_m$.

In the first case, we solve system (1.3) for $u_1^\circ, \ldots, u_k^\circ$. The resulting equalities

$$P_\nu^*(h_1^*,\dots,h_m^*) \equiv f_\nu(0,u_{k+1}^\circ(h_1^*,\dots,h_m^*),\dots,u_l^\circ(h_1^*,\dots,h_m^*),T^*) = 0 \quad (\nu = n-k+1,\dots,n) \tag{1.9}$$

must hold for the parameters $h_1^*, \dots, h_m^*$ of a generating solution. If at the same time one of the conditions

$$\mathrm{Ra}^* = \operatorname{rank}\left\| \frac{\partial P_\nu(h_1^*,\dots,h_m^*,T^*)}{\partial h_j^*} \right\| = n-k \tag{1.10}$$

$$\mathrm{Ra}^* = n-k-1, \ \mathrm{Ra}_1^* = \operatorname{rank}\left\| \frac{\partial P_\nu(h_1^*,\dots,h_m^*,T^*)}{\partial h_j^*}, \ \frac{\partial P_\nu(h_1^*,\dots,h_m^*,T^*)}{\partial T^*} \right\| = n-k \tag{1.11}$$

also holds, then (1.2) is a generating solution. If T is independent of $h_1, \dots, h_m$ for the family under consideration, and (1.10) is satisfied, then the generating system, together with solution (1.2), has a family of T generating solutions including (1.2).

In the second case we solve Eqs (1.3) for $u_1^\circ, \dots, u_k^\circ$. As a result we obtain $n-k-1$ equations of the form (1.8) to determine the parameters of a generating solution for which the rank of the Jacobian must equal $n-k-1$. This last condition is equivalent to the requirement that $\mathrm{Ra}_1^* = n-k$ in (1.11).

2. CONTINUABILITY CONDITIONS

We first consider a periodic system. Equations (1.7) are the result of eliminating $\Delta u_1^\circ, \dots, \Delta u_k^\circ$ from the equations

$$\sum_{j=1}^{l} F_{sj}^* \Delta u_j^\circ + Q_s^* \mu = 0 \quad (s = 1,\dots,n) \tag{2.1}$$

When $\mu = 0$ the partial derivatives $\partial u_\alpha/\partial \upsilon_j^\circ$, $\partial u_\beta/\partial u_j^\circ$ $(\alpha, j = 1, \dots, l;\ \beta = 1, \dots, n)$ satisfy equations in variations set up for the generating system in the neighbourhood of solution (1.2). These equations are

$$\begin{aligned} \mathbf{x}^\cdot &= \mathbf{A}^-(t)\mathbf{x} + \mathbf{A}^+(t)\mathbf{y} \\ \mathbf{y}^\cdot &= \mathbf{B}^+(t)\mathbf{x} + \mathbf{B}^-(t)\mathbf{y}; \quad \mathbf{x} \in \mathbb{R}^l, \quad \mathbf{y} \in \mathbb{R}^n \end{aligned} \tag{2.2}$$

where the plus (minus) sign denotes a 2π-periodic matrix with even (odd) functions.

The fundamental solution matrix of system (2.2) with identity matrix of initial data (at $t = \tau$) is written in the form

$$\mathbf{S}(t,\tau) = \left\| \begin{matrix} \mathbf{x}^+(t,\tau) & \mathbf{x}^-(t,\tau) \\ \mathbf{y}^-(t,\tau) & \mathbf{y}^+(t,\tau) \end{matrix} \right\|, \quad \mathbf{S}(\tau,\tau) = \mathbf{I}_{l+n}$$

where $\mathbf{I}_{l+n}$ is the identity matrix of order $l + n$. Then [2]

$$\mathbf{x}^\pm(-t,0) = \pm\mathbf{x}^\pm(t,0), \quad \mathbf{y}^\pm(-t,0) = \pm\mathbf{y}^\pm(t,0)$$

When $t = 0$ we have $u_\alpha = u_\alpha^\circ$, $\upsilon_\beta = 0$ $(\alpha = 1, \dots, l;\ \beta = 1, \dots, n)$. Therefore

$$\left(\frac{\partial u_\alpha}{\partial u_j^\circ}\right)_0 = \delta_{\alpha j}, \quad \left(\frac{\partial \upsilon_\beta}{\partial u_j^\circ}\right)_0 = 0 \quad (\alpha, j = 1,\dots,l;\ \beta = 1,\dots,n) \tag{2.3}$$

($\delta_{\alpha j}$ is the Kronecker delta). The solution of system (2.2) with initial data (2.3) is given by matrices $\mathbf{x}^+(t, 0)$, $\mathbf{y}^-(t, 0)$, where $\mathbf{y}^-(t, 0) = \| \partial\upsilon_\beta/\partial u_j^\circ \|$. Consequently, in (2.1) we have $\| F_{sj}^* \| = \mathbf{y}^-(\pi, 0)$. If $\operatorname{rank}\mathbf{y}^-(\pi, 0) = n - k$, then $l - k$ columns in the matrix $\mathbf{y}^-(\pi, 0)$ may be equated to zero. Using the fact that a linear combination of solutions of a linear system is also a solution, we conclude that exactly $l - k$ solutions exist such that at $t = \pi$ the vector $\mathbf{y}^-$ vanishes. If we now note that $\mathbf{y}^-(0, 0) = \mathbf{0}$, we can quickly establish the existence of $l - k$ 2π-periodic particular solutions of system (2.2), symmetric relative to the set $\mathbf{M}_1^* = \{\mathbf{x}, \mathbf{y} : \mathbf{y} = \mathbf{0}\}$. Then the matrix $\mathbf{x}^+(2\pi, 0)$ has $l - k$ simple eigenvalues equal to 1. Thus the matrix $\mathbf{y}^+(2\pi, 0)$ has $n - k$ such eigenvalues, and system (2.2) admits of $n - k^*$ ($k^* \le k$) 2π-periodic motions symmetric relative to the set $\mathbf{M}_2^* = \{\mathbf{x}, \mathbf{y} : \mathbf{x} = \mathbf{0}\}$.

Let

$$\mathbf{S}_1(t) = \begin{Vmatrix} \mathbf{p}^+(t) & \mathbf{p}^-(t) \\ \mathbf{q}^-(t) & \mathbf{q}^+(t) \end{Vmatrix}, \quad \mathbf{S}_1(0) = \mathbf{I}_{l+n}$$

be a fundamental solution matrix of the system conjugate to system (2.2). Then $\mathbf{S}_1(t)$ contains $l - k^*$ 2π-periodic solutions $\{p_{\alpha\lambda}^+(t), q_{\beta\lambda}^-(t)\}$ ($\lambda = 1, \ldots, l - k^*$), symmetric over the set $\{\mathbf{p}, \mathbf{q}: \mathbf{q} = \mathbf{0}\}$, and $n - k$ 2π-periodic solutions $\{p_{\alpha\nu}^-(t), q_{\beta\nu}^+(t)\}$ ($\nu = 1, \ldots, n - k$), symmetric over the set $\{\mathbf{p}, \mathbf{q}: \mathbf{q} = \mathbf{0}\}$. The relation between the solutions of conjugate systems yields the following equalities

$$\begin{aligned}
p_{j\lambda}^+(\tau) &= \sum_{\alpha=1}^{l} x_{\alpha j}^+(t,\tau) p_{\alpha\lambda}^+(t) + \sum_{\beta=1}^{n} y_{\beta j}^-(t,\tau) q_{\beta\lambda}^-(t) \\
q_{s\lambda}^-(\tau) &= \sum_{\alpha=1}^{l} x_{\alpha j}^-(t,\tau) p_{\alpha\lambda}^+(t) + \sum_{\beta=1}^{n} y_{\beta s}^+(t,\tau) q_{\beta\lambda}^-(t) \\
p_{j\nu}^-(\tau) &= \sum_{\alpha=1}^{l} x_{\alpha j}^+(t,\tau) p_{\alpha\nu}^-(t) + \sum_{\beta=1}^{n} y_{\beta j}^-(t,\tau) q_{\beta\nu}^+(t) \\
q_{s\nu}^+(\tau) &= \sum_{\alpha=1}^{l} x_{\alpha s}^-(t,\tau) p_{\alpha\nu}^-(t) + \sum_{\beta=1}^{n} y_{\beta s}^+(t,\tau) q_{\beta\nu}^+(t) \\
&(j = 1,\ldots,l;\ s = 1,\ldots,n;\ \lambda = 1,\ldots,l - k^*;\ \nu = 1,\ldots,n - k)
\end{aligned}$$

From these we derive the relations

$$p_{j\nu}^-(0) = \sum_{\beta=1}^{n} y_{\beta j}^-(\pi,0) q_{\beta\nu}^+(\pi) = 0, \quad p_{j\nu}^-(\tau) = \sum_{\beta=1}^{n} y_{\beta j}^-(\pi,\tau) q_{\beta\nu}^+(\pi)$$

$$q_{s\nu}^+(\tau) = \sum_{\beta=1}^{n} y_{\beta s}^+(\pi,\tau) q_{s\nu}^+(\pi)$$

needed to eliminate $\Delta u_1^\circ, \ldots, \Delta u_k^\circ$ from (2.1).

From the complete solution of our problem, we note that the partial derivatives of $\mathbf{u}$ and $\mathbf{v}$ with respect to μ form a particular solution of the linear system

$$\begin{aligned}
\mathbf{x}^{\cdot} &= \mathbf{A}^-(t)\mathbf{x} + \mathbf{A}^+(t)\mathbf{y} + \mathbf{U}_1(0, \varphi, \psi, t) \\
\mathbf{y}^{\cdot} &= \mathbf{B}^+(t)\mathbf{x} + \mathbf{B}^-(t)\mathbf{y} + \mathbf{V}_1(0, \varphi, \psi, t)
\end{aligned} \tag{2.4}$$

with zero initial data [4]. Therefore

$$\frac{\partial v_s}{\partial \mu} = \int_0^t \left[\sum_{\varkappa=1}^{l} U_{1\varkappa}^*(\mathbf{h}^*, \tau) y_{s\varkappa}^-(t,\tau) + \sum_{\nu=1}^{n} V_{1\nu}^*(\mathbf{h}^*, \tau) y_{s\nu}^+(t,\tau) \right] d\tau$$

where $\mathbf{U}_1^*(\mathbf{h}^*, \tau)$ and $\mathbf{V}_1^*(\mathbf{h}^*, \tau)$ denote the result of substituting $\mu = 0$, $\mathbf{u} = \varphi(\tau)$, $\mathbf{v} = \psi(\tau)$, $t = \tau$ into the functions $\mathbf{U}_1$ and $\mathbf{V}_1$, respectively. Since solution (1.2) belongs to some family parametrized by $\mathbf{h}$ and corresponds to the values $\mathbf{h}^*$, it follows that these functions also depend on $\mathbf{h}^*$.

Conditions (2.1) now become

$$\sum_{j=1}^{l} y_{\beta j}^-(\pi,0)\Delta u_j^\circ + \mu \int_0^\pi \left[\sum_{\varkappa=1}^{l} U_{1\varkappa}^*(\mathbf{h}^*, \tau) y_{\beta\varkappa}^-(\pi,\tau) + \sum_{\delta=1}^{n} V_{1\delta}^*(\mathbf{h}^*, \tau) y_{\beta\delta}^+(\pi,\tau) \right] d\tau = 0 \quad (\beta = 1,\ldots,n)$$

Multiplying these equalities by $q_{\beta\nu}^+(\pi)$ and summing over β, we get

$$P_\nu(h_1^*,\ldots,h_m^*)\equiv\int_0^\pi\left[\sum_{\varkappa=1}^{l}U_{1\varkappa}^*(\mathbf{h}^*,\tau)p_{\varkappa\nu}^-(\tau)+\sum_{\delta=1}^{n}V_{1\delta}^*(\mathbf{h}^*,\tau)q_{\delta\nu}^+(\tau)\right]d\tau=0\quad(\nu=1,\ldots,n-k)\tag{2.5}$$

Theorem 1. Let (1.2) be a 2π-periodic reversible system (1.1) corresponding to parameter values $h_1^*, \ldots, h_m^*$, and let $\mathrm{Ra} = k < n$. Then it is a generating solution, provided that equalities (2.5) hold and $\mathrm{Ra}^* = n - k$.

Example 1. Resonance oscillations in the Duffing problem [4, p. 32]. The Duffing equation in this case

$$x^{\cdot\cdot}+x=\mu(\lambda\sin t+ax+\gamma x^3);\quad \lambda,a,\gamma=\mathrm{const}$$

is reversible with fixed set $\{x,x^{\cdot}:x = 0\}$. At $\mu = 0$ it has a one-parameter family (parameter h) of symmetric periodic solutions $x = h\sin t$. In system (1.1), therefore, we have $l = n = 1$, $\mathrm{Ra} = 0$. To determine whether a periodic solution exists when $\mu \neq 0$, we set up Eq. (2.5)

$$\lambda+ah^*+0{,}75\gamma h^{*3}=0$$

By Theorem 1, for each simple root h^* of this equation we have a generating solution of a linear oscillator. Note that only symmetric periodic motions are defined for the Duffing equation [4].

In the autonomous system (1.1), the equations defining the parameters of a generating solution are

$$\sum_{j=1}^{l}F_{sj}^*\Delta u_j^\circ+G_s^*\Delta T+Q_s^*\mu=0\quad(s=1,\ldots,n)$$

If $\mathrm{Ra}^* = \mathrm{Ra}_1^* = n-k$, then G_s^* will be linear combinations of F_{sj}^*. Hence it follows from the equalities

$$p_{j\nu}^-(0)=\sum_{\beta=1}^{n}y_{\beta j}^-(T^*,0)q_{\beta\nu}^+(T^*)=0\quad(j=1,\ldots,l;\ \nu=1,\ldots,n-k)$$

that

$$\sum_{\beta=1}^{n}G_\beta^*q_{\beta\nu}^+(T^*)=0\quad(\nu=1,\ldots,n-k)$$

As a result we obtain the following equations for the parameters of a generating solution

$$P_\nu^*(h_1^*,\ldots,h_m^*,T^*)\equiv\int_0^{T^*}\left[\sum_{\varkappa=1}^{l}U_{1\varkappa}^*(\mathbf{h}^*,\tau)p_{\varkappa\nu}^-(\tau)+\sum_{\delta=1}^{n}V_{1\delta}^*(\mathbf{h}^*,\tau)q_{\delta\nu}^+(\tau)\right]d\tau=0\quad(\nu=1,\ldots,n-k)\tag{2.6}$$

In the case $\mathrm{Ra}^* = k$, $\mathrm{Ra}_1^* = k + 1 < n$ we have

$$\sum_{\beta=1}^{n}G_\beta^*q_{\beta\nu}^+(T^*)=\sum_{\beta=1}^{n}V_{0\beta}^*(\mathbf{h}^*,T^*)q_{\beta\nu}^+(T^*)$$

(where $\mathbf{V}_0^*$ is the result of substituting solution (1.2) into the function $\mathbf{V}_0$) and the equations for determining γ ($\Delta T = \gamma\mu$) and $h_1^*, \ldots, h_m^*$ are

$$P_\nu^*(h_1^*,\ldots,h_m^*,T^*,\gamma)=\gamma\sum_{\beta=1}^{n}V_{0\beta}^*(\mathbf{h}^*,T^*)q_{\beta\nu}^+(T^*)+$$

$$+\int_0^{T^*}\left[\sum_{\varkappa=1}^{l}U_{1\varkappa}^*(\mathbf{h}^*,\tau)p_{\varkappa\nu}^-(\tau)+\sum_{\delta=1}^{n}V_{1\delta}^*(\mathbf{h}^*,\tau)q_{\delta\nu}^+(\tau)\right]d\tau=0\quad(\nu=1,\ldots,n-k)\tag{2.7}$$

Theorem 2. Let (1.2) be a $2T^*\pi$-periodic solution autonomous system (1.1) corresponding to a parameter values $h_1^*, \ldots, h_m^*$, and let $\mathrm{Ra} = k < n$. Then, if $\mathrm{Ra}_1 = k$, it is a generating solution provided that equalities (2.6) hold and $\mathrm{Ra}^* = n-k$ or $\mathrm{Ra}^* = n-k-1$, $\mathrm{Ra}_1^* = n-k$. If $\mathrm{Ra} = k$, $\mathrm{Ra}_1 = k + 1 < n$, the parameters of the generating solution and the correction ΔT are determined from system (2.7) if

$$\mathrm{Ra}_1^* = \left\| \frac{\partial P_\nu^*(h_1^*,\ldots,h_m^*,T^*,\gamma)}{\partial h_\beta^*}, \ \sum_{\beta=1}^{n} V_{0\beta}^*(\mathbf{h}^*,T^*)q_{\beta\nu}^+(T^*) \right\| = n-k$$

3. THE CHOICE OF A GENERATING SYSTEM

The problem of continuing symmetric periodic motions in non-structurally stable cases may be reduced to the structurally stable case by suitable choice of a generating system which depends on a small parameter. Examples of such solutions of the problem are known [5, 6].

Let us consider an autonomous or 2π-periodic reversible system

$$\begin{aligned} \mathbf{u}^{\cdot} &= \mathbf{U}_0(\varepsilon,\mathbf{u},\mathbf{v},t)+\mu\mathbf{U}_1 \quad (\varepsilon,\mu,\mathbf{u},\mathbf{v},t) \\ \mathbf{v}^{\cdot} &= \mathbf{V}_0(\varepsilon,\mathbf{u},\mathbf{v},t)+\mu\mathbf{V}_1 \quad (\varepsilon,\mu,\mathbf{u},\mathbf{v},t); \quad \mathbf{u}\in\mathbb{R}^l, \ \ \mathbf{v}\in\mathbb{R}^n \quad (l\geqslant n) \end{aligned} \tag{3.1}$$

with fixed set $\mathbf{M} = \{\mathbf{u}, \mathbf{v} : \mathbf{v} = \mathbf{0}\}$ and small parameters ε and μ. Suppose that for $\mu = 0$ system (3.1) has a $2T^*$-periodic solution, symmetric relative to $\mathbf{M}$

$$\mathbf{u} = \varphi(\varepsilon,t), \quad \mathbf{v} = \psi(\varepsilon,t); \quad \varphi(\varepsilon,-t) = \varphi(\varepsilon,t), \quad \psi(\varepsilon,-t) = -\psi(\varepsilon,t) \tag{3.2}$$

where $T^*(\varepsilon) = \pi$ in the case of a 2π-periodic system, and when $\varepsilon = 0$ solution (3.2) belongs to an m-family. It is interesting to determine for what class of functions $\mu = \mu(\varepsilon)$ the question of whether system (3.1) admits of a symmetric periodic motion can be solved only by the generating system obtained from (3.1) by putting $\mu = 0$.

Necessary and sufficient conditions for a symmetric solution to be $2T$-periodic have the form (1.3), but now the left-hand sides also depend on ε. Put

$$\varphi(\varepsilon,0) = \mathbf{u}^*, \quad \Delta u_j^\circ = u_j^\circ - u_j^* \quad (j=1,\ldots,l), \quad \Delta T = T - T^*(\varepsilon)$$

and write (1.3) as

$$\sum_{j=1}^{l} [F_{sj}^*(\varepsilon)+F_{sj}]\Delta u_j^\circ + [G_s^*(\varepsilon)+G_s]\Delta T + \mu v_{1s}(\varepsilon,\mu,\Delta u^\circ,\Delta T) = 0 \quad (s=1,\ldots,n) \tag{3.3}$$

where the functions F_{sj}, G_s vanish for $\Delta u^\circ = 0$, $\Delta T = 0$.

Let $\mathrm{Ra} = \mathrm{rank}\| F_{sj}^*(0) \| = k < n$. Then we may assume without loss of generality that

$$F_{sj}^*(\varepsilon) = \varepsilon[a_{sj} + F_{sj}^\circ(\varepsilon)] \quad (s = n-k+1,\ldots,n; \ j=1,\ldots,l)$$

where the coefficients a_{sj} do not depend on ε and the functions $F_{sj}^\circ(\varepsilon)$ vanish when $\varepsilon = 0$. In the case of a 2π-periodic system we have $\Delta T = 0$ and the last $n-k$ equations in (3.3) take the form

$$\sum_{j=1}^{l} \left[a_{sj} + F_{sj}^\circ(\varepsilon) + \frac{1}{\varepsilon}F_{sj} \right]\Delta u_j^\circ + \frac{\mu}{\varepsilon} v_{1s}(\varepsilon,\mu,\Delta u^\circ,\Delta T) = 0 \quad (s=n-k+1,\ldots,n)$$

Therefore, if $\mu = o(\varepsilon)$ as $\varepsilon \to 0$, system (3.3) is solvable for n of the quantities $\Delta u_1^\circ, \ldots, \Delta u_l^\circ$ provided that $\mathrm{rank}\| F_{sj}^*(\varepsilon) \| = n$, taking into account terms linear in ε.

An autonomous system (3.1) is investigated in the same way.

Put

$$r = \mathrm{rank}\left\| \begin{matrix} F_{sj}^\circ(0) \\ a_{sj} \end{matrix} \right\|, \quad r_1 = \mathrm{rank}\left\| \begin{matrix} F_{sj}^\circ(0) & G_s^*(0) \\ a_{sj} & b_s \end{matrix} \right\| \tag{3.4}$$

$$F_{sj}^\circ(0) = \left(\frac{\partial v_s(\varepsilon,\mu,\mathbf{u}^\circ,T)}{\partial u_j^\circ} \right)_0, \quad G_s^*(0) = \left(\frac{\partial v_s(\varepsilon,\mu,\mathbf{u}^\circ,T)}{\partial T} \right)_0$$

$$a_{sj} = \left(\frac{\partial^2 v_s(\varepsilon,\mu,\mathbf{u}^\circ,T}{\partial u_j^\circ \partial\varepsilon}\right)_0, \quad b_s = \left(\frac{\partial^2 v_s(\varepsilon,\mu,\mathbf{u}^\circ,T)}{\partial T\partial\varepsilon}\right)_0$$

Then the following theorem holds.

Theorem 3. Suppose that when $\mu = 0$ system (3.1) has a symmetric periodic solution (3.2) which, when $\varepsilon = 0$, belongs to an m-family ($m \geqslant n - k$). If also $r = n$, then system (3.1) has a periodic solution, symmetric relative to the fixed set **M**, for any perturbations such that $\mu = o(\varepsilon)$ as $\varepsilon \to 0$. In autonomous system (3.1), the existence of a periodic solution for perturbations satisfying this condition is also guaranteed when $r_1 = n$.

Remark. In Sections 1 and 2 it was the part of system (1.1) linear in μ that was taken as the generating system for (3.1); however, no symmetric periodic solution of the latter is known. Theorems 1 and 2 enable us to solve the question of whether (1.1) has a periodic solution without having to calculate such a symmetric solution.

Example 2. *Generalization of Hill's problem.* The planar orbits of a passively gravitating point in the neighbourhood of one of the main bodies in the circular restricted three-body problem are described by a reversible system [6]

$$x^{\cdot\cdot} - 2my^{\cdot} + kx\rho^{-3} = m^2X(m,x,y) \tag{3.5}$$

$$y^{\cdot\cdot} + 2mx^{\cdot} + ky\rho^{-3} = m^2Y(m,x,y), \quad \rho^2 = x^2 + y^2, \quad k = \text{const}$$

with fixed set $\mathbf{M} = \{x, y, x^{\cdot}, y : y = 0, x^{\cdot} = 0\}$ and small parameter m. This system contains Hill's problem concerning the motion of the Moon [7], which is obtained by setting $X = 3x$, $Y = 0$ in (3.5).

When $m = 0$ system (3.5) has a circular particular solution, but it is not a generating solution. Lyapunov [5], who presented the first rigorous proof of existence for nearly circular orbits in Hill's problem, used a generating system which depended on the parameter m. In so doing he verified the conditions for continuability taking into account terms linear in m.

As the generating system for problem (3.5), let us take the system

$$x^{\cdot\cdot} - 2my^{\cdot} + kx\rho^{-3} - m^2x = 0, \quad y^{\cdot\cdot} + 2myx^{\cdot\cdot} + ky\rho^{-3} - m^2y = 0 \tag{3.6}$$

which admits of circular orbits

$$x = a\cos\omega t, \quad y = a\sin\omega t, \quad ka^{-3} = (\omega + m)^2 \tag{3.7}$$

symmetric relative to the set **M** and belonging to a family of elliptical orbits. To answer the question of whether system (3.5) has orbits close to (3.7), we set

$$x + iy = a\exp(i\omega t)(1 + \alpha + i\beta)$$

Then, in the linear approximation with respect to α and β, we have

$$\alpha^{\cdot\cdot} + (\omega + m)^2\alpha - 2(\omega + m)\gamma = 0, \quad \beta^{\cdot} = \gamma - 2(\omega + m)\alpha, \quad \gamma^{\cdot} = 0 \tag{3.8}$$

and corresponding to the fixed set **M** we have a set $\{\alpha, \beta, \alpha^{\cdot}, \gamma: \beta = 0, \alpha^{\cdot} = 0\}$. Hence the zero roots of the characteristic equation do not obstruct continuation with respect to the parameter [2]. The remaining pair of roots $\pm i(\omega + m)$, evaluated to terms linear in m, does not contain the critical roots, which are $\pm ip\omega$ ($p \in \mathbb{N}$). Consequently, we have $r = n = 2$ in (3.4), and for small $m \neq 0$ system (3.5) has periodic orbits, symmetric relative to **M**, which are close to the circular orbits (3.7).

4. REDUCTION TO A QUASI-LINEAR SYSTEM

Non-structurally stable cases also arise when it is not known in advance if solution (1.2) of the generating system belongs to some family. Moreover, the results of Sections 1 and 2 enable us to establish the existence of periodic motions only in an $O(\mu)$-neighbourhood of a generating solution, though we know [4, 8, 9] of motions that differ from generating motions by quantities $O(\mu^{1/s})$, $s \in \mathbb{N}$.

These difficulties may be solved by reduction to the problem of extending solutions of a linear system with respect to a parameter. The approach proposed below is also applicable, with obvious corrections, to systems of a general form.

Suppose it is known that the generating system obtained from (1.1) with $\mu = 0$ admits of a 2π-periodic solution (1.2). Put

$$\mathbf{u} = \boldsymbol{\varphi}(t) + \mathbf{x}, \quad \mathbf{v} = \boldsymbol{\psi}(t) + \mathbf{y}$$

Then we obtain a 2π-periodic system for the variables $\mathbf{x}$, $\mathbf{y}$

$$\begin{aligned} \mathbf{x}^{\cdot} &= \mathbf{A}^-(t)\mathbf{x} + \mathbf{A}^+(t)\mathbf{y} + \mathbf{X}(\mathbf{x},\mathbf{y},t) + \mu\mathbf{U}_1 \quad (\mu, \boldsymbol{\varphi}+\mathbf{x}, \boldsymbol{\psi}+\mathbf{y}, t) \\ \mathbf{y}^{\cdot} &= \mathbf{B}^+(t)\mathbf{x} + \mathbf{B}^-(t)\mathbf{y} + \mathbf{Y}(\mathbf{x},\mathbf{y},t) + \mu\mathbf{V}_1 \quad (\mu, \boldsymbol{\varphi}+\mathbf{x}, \boldsymbol{\psi}+\mathbf{y}, t) \end{aligned} \tag{4.1}$$

which is reversible with fixed set $\mathbf{M}_1 = \{\mathbf{x}, \mathbf{y} : \mathbf{y} = \mathbf{0}\}$ [10]. Obviously, the linear part of system (4.1) with respect to $\mathbf{x}$, $\mathbf{y}$ is identical with (2.2) when $\mu = 0$.

We now apply the change of variables $(\mathbf{x}, \mathbf{y}) \to (\varepsilon\mathbf{x}, \varepsilon\mathbf{y})$, $\varepsilon = \mu^\sigma$, $0 < \sigma \leqslant 1$. This gives system (1.1) the form

$$\begin{aligned} \mathbf{x}^{\cdot} &= \mathbf{A}^-(t)\mathbf{x} + \mathbf{A}^+(t)\mathbf{y} + \varepsilon^\lambda[\mathbf{X}_0(\mathbf{x},\mathbf{y},t) + \varepsilon\mathbf{X}_1(\varepsilon,\mathbf{x},\mathbf{y},t)] + \mu^{1-\sigma}\mathbf{U}_1(\mu, \boldsymbol{\varphi}+\varepsilon\mathbf{x}, \boldsymbol{\psi}+\varepsilon\mathbf{y}, t) \\ \mathbf{y}^{\cdot} &= \mathbf{B}^+(t)\mathbf{x} + \mathbf{B}^-(t)\mathbf{y} + \varepsilon^\lambda[\mathbf{Y}_0(\mathbf{x},\mathbf{y},t) + \varepsilon\mathbf{Y}_1(\varepsilon,\mathbf{x},\mathbf{y},t)] + \mu^{1-\sigma}\mathbf{V}_1(\mu, \boldsymbol{\varphi}+\varepsilon\mathbf{x}, \boldsymbol{\psi}+\varepsilon\mathbf{y}, t) \end{aligned} \tag{4.2}$$

($\lambda \geqslant 1$, $\lambda \in \mathbb{N}$). If $\operatorname{Ra} = k < n$, the new generating system obtained from (4.2) by putting $\mu = 0$, $\varepsilon = 0$ admits of exactly $l - k$ 2π-periodic solutions symmetric relative to $\mathbf{M}_1$

$$x_j = \varphi^*_{j\alpha}(t), \quad y_s = \psi^*_{s\alpha}(t) \quad (j = 1,\dots,l; \; s = 1,\dots,n; \; \alpha = 1,\dots,l-k)$$

Thus, this system has an $(l-k)$-family of parameters $h_1, \dots, h_{l-k}$ of periodic solutions

$$x_j = \varphi^*_j(\mathbf{h},t) \equiv \sum_{\alpha=1}^{l-k} h_\alpha \varphi^*_{j\alpha}(t), \quad y_s = \psi^*_s(\mathbf{h},t) \equiv \sum_{\alpha=1}^{l-k} h_\alpha \psi^*_{s\alpha}(t) \tag{4.3}$$

Suppose that not all the integrals (2.5) vanish for solution (1.2) under consideration. Then we set $\sigma = 1/(1 + \lambda)$ and consider the equations

$$P_\nu(\mathbf{h}) \equiv \int_0^\pi \left[\sum_{\alpha=1}^{l} X^*_\alpha(\mathbf{h},\tau) p^-_{\alpha\nu}(\tau) + \sum_{\beta=1}^{n} Y^*_\beta(\mathbf{h},\tau) q^+_{\beta\nu}(\tau) \right] d\tau = 0 \quad (\nu = 1,\dots,n-k) \tag{4.4}$$

$$\mathbf{X}^*(\mathbf{h},t) \equiv \mathbf{X}_0(\boldsymbol{\varphi}^*(\mathbf{h},t), \; \boldsymbol{\psi}^*(\mathbf{h},t),t) + \mathbf{U}_1(0, \boldsymbol{\varphi}(t), \boldsymbol{\psi}(t), t)$$

$$\mathbf{Y}^*(\mathbf{h},t) \equiv \mathbf{Y}_0(\boldsymbol{\varphi}^*(\mathbf{h},t), \; \boldsymbol{\psi}^*(\mathbf{h},t),t) + \mathbf{V}_1(0, \boldsymbol{\varphi}(t), \boldsymbol{\psi}(t), t)$$

where the functions $p^-_{\alpha\nu}(\tau)$, $q^+_{\beta\nu}(\tau)$ have the same meaning as in (2.5). Applying the results of Sections 1 and 2 to system (4.2), we obtain the following theorem.

Theorem 3. To each root $\mathbf{h}^*$ of Eq. (4.4) satisfying the condition

$$\operatorname{rank} \left\| \frac{\partial P_\nu(h^*_1,\dots,h^*_{l-k})}{\partial h^*_j} \right\| = n - k \tag{4.5}$$

there corresponds a 2π-periodic solution, symmetric relative to $\mathbf{M}_1$, of system (1.1)

$$\mathbf{u} = \boldsymbol{\varphi}(t) + \mu^\sigma \boldsymbol{\varphi}^*(\mathbf{h}^*,t) + o(\mu^\sigma), \quad \mathbf{v} = \boldsymbol{\psi}(t) + \mu^\sigma \boldsymbol{\psi}^*(\mathbf{h}^*,t) + o(\mu^\sigma), \quad \sigma = \frac{1}{1+\lambda}$$

which is identical with the generating solution (1.2) at $\mu = 0$.

The following example indicates a possible generalization of Theorem 3.

Example 3. The system of equations

$$x^{\cdot\cdot}+x=(\alpha x^4+y^{\cdot 2}+\mu a)\cos t,\ \ y^{\cdot\cdot}+y=(\beta x^5+x^3y+\mu b)\cos t \tag{4.6}$$

(α, β, a, b = const) is a reversible system of type (4.1) with fixed set $\{x, y, x^{\cdot}, y^{\cdot} : x^{\cdot} = 0, y^{\cdot} = 0\}$. We make the change of variables $(x, y) \to (\varepsilon x, \varepsilon^2 y)$, $\varepsilon^5 = \mu$ in (4.6). This gives

$$x^{\cdot\cdot}+x=\varepsilon^3(\alpha x^4+y^{\cdot 2}+\varepsilon a)\cos t,\ \ y^{\cdot\cdot}+y=\varepsilon^3(\beta y^5+x^3y+b)\cos t \tag{4.7}$$

When $\varepsilon = 0$ system (4.7) admits of two families of symmetric periodic motions: $x = A\cos t, y = B\cos t$. Hence Eqs (4.4) have the form

$$5\alpha A^4+6B^2=0,\ \ 5A^3B+8b=0$$

and when $\alpha < 0$ these equations have simple solutions equal, for example, to $A = \mp 1$, $B = \pm 1$ for $\alpha = -6/5$, $b = 8/5$. Consequently, for these parameter values system (4.6) has periodic solutions

$$x=\mp\mu^{1/5}\cos t+o(\mu^{1/5}),\ \ y=\pm\mu^{2/5}\cos t+o(\mu^{2/5})$$

Let us assume now that solution (1.2) satisfies condition (2.5). In that case system (2.4) admits of a family of symmetric periodic solutions

$$x_j=\varphi_j^*(\mathbf{h},t)+\theta_j(t),\quad y_s=\psi_s^*(\mathbf{h},t)+\chi_s(t);\quad j=1,\dots,l;\ s=1,\dots,n \tag{4.8}$$

Set $\sigma = 1$ in system (4.2). Then in each of the two possible cases

$$1)\ \lambda>1,\ \mathbf{X}^*\equiv\mathbf{X}_\lambda\equiv\left[\frac{\partial\mathbf{U}_1}{\partial\mu}+\frac{\partial\mathbf{U}_1}{\partial\mathbf{u}}\mathbf{x}+\frac{\partial\mathbf{U}_1}{\partial\mathbf{v}}\mathbf{y}\right]_*,\quad \mathbf{Y}^*\equiv\mathbf{Y}_\lambda\equiv\left[\frac{\partial\mathbf{V}_1}{\partial\mu}+\frac{\partial\mathbf{V}_1}{\partial\mathbf{u}}\mathbf{x}+\frac{\partial\mathbf{V}_1}{\partial\mathbf{v}}\mathbf{y}\right]_*$$

$$2)\ \lambda=1,\ \mathbf{X}^*\equiv\mathbf{X}_\lambda+[\mathbf{X}_0(\mathbf{x},\mathbf{y},t)]_*,\quad \mathbf{Y}^*\equiv\mathbf{Y}_\lambda+[\mathbf{Y}_0(\mathbf{x},\mathbf{y},t)]$$

(the asterisk on the bracketed expressions means that the derivatives are evaluated at $\mu = 0$, $\mathbf{u} = \varphi(t)$, $\mathbf{v} = \psi(t)$, and $\mathbf{x}$, $\mathbf{y}$ are replaced by (4.8)) the validity of conditions (4.4) and (4.5) guarantees the existence of solutions of system (1.1)

$$\mathbf{u}=\boldsymbol{\varphi}(t)+\mu[\boldsymbol{\varphi}^*(\mathbf{h}^*,t)+\boldsymbol{\theta}(t)]+o(\mu),\ \ \mathbf{v}=\boldsymbol{\psi}(t)+\mu[\boldsymbol{\psi}^*(\mathbf{h}^*,t)+\boldsymbol{\chi}(t)]+o(\mu)$$

Remark. As in Sections 1 and 2, a more detailed investigation, omitted here, can be carried out for autonomous systems (1.1).

5. A QUASI-LINEAR SYSTEM

The theory of oscillations for a quasi-linear reversible system is constructed in the same way as for a system of general form [4]. When that is done, however, one must consider symmetric periodic solutions of the linear system and use the results of Section 2. Omitting these constructions, we will confine our attention below to one result for an autonomous reversible system

$$\begin{aligned}&\mathbf{x}^{\cdot}=\mathbf{A}\mathbf{y}+\mu\mathbf{X}(\mu,\mathbf{x},\mathbf{y})\\&\mathbf{y}^{\cdot}=\mathbf{B}\mathbf{x}+\mu\mathbf{Y}(\mu,\mathbf{x},\mathbf{y});\quad \mathbf{x}\in\mathbb{R}^l,\quad \mathbf{y}\in\mathbb{R}^n\ (l\geqslant n)\end{aligned} \tag{5.1}$$

($\mathbf{A}$ and $\mathbf{B}$ are constant matrices) with fixed set $\{\mathbf{x}, \mathbf{y} : \mathbf{y} = \mathbf{0}\}$; the result in question has no analogue in a system of general form.

The matrix $\mathbf{A}$ has at most $l-n$ linearly independent rows, and an elementary transformation reduces $\mathbf{A}$ to a form in which $l - n$ rows are filled with zeros. Let ξ denote a variable corresponding to these rows. Then the equation for $\mathbf{y}$ is

$$\mathbf{y}^{\cdot}=\mathbf{B}_*\mathbf{x}+\mathbf{B}_1\boldsymbol{\xi}+\mu\mathbf{Y}(\mu,\boldsymbol{\xi},\mathbf{x},\mathbf{y})\quad (\boldsymbol{\xi}\in\mathbb{R}^{l-n},\ \mathbf{x}\in\mathbb{R}^n)$$

Assuming that $\det \mathbf{B}_* \neq 0$, we now replace the vector $\mathbf{x}$ by $\mathbf{x} + \mathbf{B}_*^{-1}\mathbf{B}_1\boldsymbol{\xi}$. After all the necessary algebra, system (5.1) takes the form

$$\begin{aligned}
&\xi^{\cdot} = \mu\Xi(\mu, \xi, \mathbf{x}, \mathbf{y}) \\
&\mathbf{x}^{\cdot} = \mathbf{A}_{*}\mathbf{y} + \mu\mathbf{X}(\mu, \xi, \mathbf{x}, \mathbf{y}) \qquad\qquad (5.2)\\
&\mathbf{y}^{\cdot} = \mathbf{B}_{*}\mathbf{x} + \mu\mathbf{Y}(\mu, \xi, \mathbf{x}, \mathbf{y}); \quad \xi \in \mathbb{R}^{l-n}, \quad \mathbf{x} \in \mathbb{R}^{n}, \quad \mathbf{y} \in \mathbb{R}^{n}
\end{aligned}$$

with fixed set $\mathbf{M}^* = \{\xi, \mathbf{x}, \mathbf{y} : \mathbf{y} = \mathbf{0}\}$.

Suppose that when $\mu = 0$ the characteristic equation of system (4.2) has a pair $\pm i\omega$ of pure imaginary roots, and no further roots of the form $\pm ip\omega$ ($p \in \mathbb{N}$). Then the generating system obtained from (4.2) by putting $\mu = 0$ admits of a unique family

$$\xi_s = h_s, \ x_j = \alpha_j h \cos \omega t, \ y_j = \beta_j h \sin \omega t \ \ (s = 1, \dots, l-n; \ j = 1, \dots, n) \tag{5.3}$$

of periodic solutions symmetric relative to $\mathbf{M}^*$. The parameters of this family are $h_1, \dots, h_{l-n}, h$, and α, β is one of the solutions of the system

$$-\boldsymbol{\alpha}\omega = \mathbf{A}_{*}\boldsymbol{\beta}, \quad \boldsymbol{\beta}\omega = \mathbf{B}_{*}\boldsymbol{\alpha}, \quad \boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)^T, \quad \boldsymbol{\beta} = (\beta_1, \dots, \beta_n)^T$$

Let us consider the question of the existence of periodic motions of system (5.1) when $\mu \neq 0$. The existence of the family of solutions (5.3) implies that Ra $< n$ in (1.5). When $\mu = 0$ the equation for ξ separates from the other equations, and the subsystem for **x** and **y** admits of only one family of periodic motions. Thus, Ra $= n - 1$. Let us calculate

$$y_j^{\cdot}(\pi / \omega) = h\omega\beta_j \ \ (j = 1, \dots, n)$$

noting that at least some of the numbers β_j must be non-zero. Without loss of generality, we may assume that only one coefficient does not vanish, say β_j; this can always be achieved by using a suitable linear transformation of the variables y_s. Hence we immediately conclude that $\text{Ra}_1 = n$ in (1.5).

Theorem 4. If rank$\mathbf{B} = n$ in the quasi-linear autonomous reversible system (5.1), then for every pair $\pm i\omega$ of roots of the characteristic equation there is a $2T(\mu)$-periodic family, with parameters $h_1, \dots, h_{l-n}, h$ of solutions symmetric relative to the set $\{\mathbf{x}, \mathbf{y} : \mathbf{y} = \mathbf{0}\}$, where $T(0) = \pi/\omega$, provided none of the other roots is equal to $\pm ip\omega$ ($p \in \mathbb{N}$).

Corollary (the Lyapunov-Bryuno–Devaney theorem). If the characteristic equation of the reversible system

$$\begin{aligned}
&\mathbf{x}^{\cdot} = \mathbf{A}\mathbf{y} + \mathbf{X}(\mathbf{x}, \mathbf{y}), \ \ \mathbf{y}^{\cdot} = \mathbf{B}\mathbf{x} + \mathbf{Y}(\mathbf{x}, \mathbf{y}); \ \ \mathbf{x} \in \mathbb{R}^{l}, \ \ \mathbf{y} \in \mathbb{R}^{n} \ \ (l \geqslant n) \\
&\mathbf{X}(\mathbf{x}, -\mathbf{y}) = -\mathbf{X}(\mathbf{x}, \mathbf{y}), \ \ \mathbf{Y}(\mathbf{x}, -\mathbf{y}) = \mathbf{Y}(\mathbf{x}, \mathbf{y})
\end{aligned}$$

(**A** and **B** are constant matrices and **X** and **Y** are non-linear terms) has a pair of pure imaginary roots, and none of the other roots is equal to $\pm ip\omega$($p \in \mathbb{N}$), rank $\mathbf{B} = n$, then the $(l - n + 1)$-family of Lyapunov periodic motions adjoins the zero equilibrium position.

Indeed, we make the change of variables $(\mathbf{x}, \mathbf{y}) \to (\mu\mathbf{x}, \mu\mathbf{y})$. Then the problem is that of continuing a periodic solution of the quasi-linear system (5.1) with respect to the parameter.

6. THREE-DIMENSIONAL PERIODIC ORBITS IN THE N-PLANET PROBLEM

Let us consider the basic problem of celestial mechanics—the motion of a mechanical system consisting of $N + 1$ point masses $\mathbf{S}, \mathbf{P}_1, \dots, \mathbf{P}_N$ that attract one another according to Newton's law. We shall assume that the mass of the main body **S** is considerably greater than the masses of the bodies $\mathbf{P}_j$; we are interested in the question of the existence of periodic orbits in this N-planet problem.

Studies of this problem are rare. Some consideration has been given [13] to planar orbits that are nearly circular and are closed in a frame of reference rotating at constant angular velocity (orbits of the first kind). The existence of symmetric periodic orbits of the first kind has been proved [14], as has that of orbits that are nearly elliptical and are closed in a stationary frame of reference (orbits of the second kind). A result concerning the existence of three-dimensional periodic orbits (orbits of the third kind) has also been formulated [3]. We now present a proof of that result, based on the theory of Sections 2 and 4, and estimate, in terms of the small parameter, the degree to which the orbit of each planet $\mathbf{P}_s$ is nearly planar.

The motion of each planet $\mathbf{P}_s$ will be considered in a frame of reference with origin at S, the main body. Then the position of $\mathbf{P}_s$ is determined by a triple of Cartesian coordinates (ξ_s, η_s, ζ_s). The equations of motion of such a system are known [15]. In the system of equations thus obtained we make the change of variables

$$x_s=\xi_s,\ \ y_s=\eta_s\cos\varphi_s+\zeta_s\sin\varphi_s,\ \ z_s=-\eta_s\sin\varphi_s+\zeta_s\cos\varphi_s\ \ (s=1,\dots,N)$$

with arbitrary parameter φ_s. This gives

$$\frac{d^2w_s}{dt^2}+\frac{k_sw_s}{r_s^3}=\mu W_s(\mathbf{w},\varphi)\qquad(s=1,\dots,N)\tag{6.1}$$

Here w_s denotes the variables x_s, y_s, z_s, k_s is the product of the gravitational constant and the total mass of the bodies S and $\mathbf{P}_s$, $\mathbf{r}_s$ is the distance from S to $\mathbf{P}_s$, μ is the ratio of the largest planetary mass to the mass of S ($\mu \ll 1$), and W_s is the right-hand sides of the equations for x_s, y_s and z_s (throughout, summation will always be performed from $j=1$ to $j=N$, where $j\neq s$)

$$X_s=\sum k_j^*\left[\frac{x_s-x_j}{r_{sj}^3}-\frac{x_j}{r_j^3}\right]$$

$$Y_s=\sum k_j^*\left\{\frac{y_s}{r_{sj}^3}-\left[\frac{1}{r_{sj}^3}-\frac{1}{r_j^3}\right][y_j\cos(\varphi_s-\varphi_j)+z_s\sin(\varphi_s-\varphi_j)]\right\}\tag{6.2}$$

$$Z_s=\sum k_j^*\left\{\frac{z_s}{r_{sj}^3}+\left[\frac{1}{r_{sj}^3}-\frac{1}{r_j^3}\right][y_j\sin(\varphi_s-\varphi_j)-z_s\cos(\varphi_s-\varphi_j)]\right\}$$

$$r_{sj}^2=(x_s-x_j)^2+(y_s-y_j)^2+(z_s-z_j)^2+2\{(y_sy_j+z_sz_j)[1-\cos(\varphi_s-\varphi_j)]-$$
$$-(y_sz_j-y_jz_s)\sin(\varphi_s-\varphi_j)]\}\quad(s,j=1,\dots,N;\ s\neq j)$$

System (6.1), (6.2) is invariant with respect to each of the transformations $(t,\mathbf{x},\mathbf{y},\mathbf{z})\to(-t,\mathbf{x},-\mathbf{y},-\mathbf{z})$ and $(t,\mathbf{x},\mathbf{y},\mathbf{z})\to(-t,-\mathbf{x},\mathbf{y},\mathbf{z})$, that is, it is linearly reversible of type (1.1), with two fixed sets $\mathbf{M}_1=(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{x}^{\cdot},\mathbf{y}^{\cdot},\mathbf{z}^{\cdot}:\mathbf{x}^{\cdot}=\mathbf{0},\mathbf{y}=\mathbf{0},\mathbf{z}=\mathbf{0}\}$ and $\mathbf{M}_2=(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{x}^{\cdot},\mathbf{y}^{\cdot},\mathbf{z}^{\cdot}:\mathbf{x}=\mathbf{0},\mathbf{y}^{\cdot}=\mathbf{0},\mathbf{z}^{\cdot}=\mathbf{0}\}$.

At $\mu=0$ system (6.1) splits into N two-body problems, each of which describes an unperturbed Kepler problem. The resulting orbits are second-order curves

$$r_s^*(\theta_s)=\frac{\lambda_s}{1+e_s\cos\theta_s},\ \ r_s^{*2}(\theta_s)\frac{d\theta_s}{dt}=c_s,\ \ \lambda_s=\frac{c_s^2}{k_s},\ \ e_s=\left(1+h_s\frac{c_s^2}{k_s^2}\right)^{1/2}\tag{6.3}$$

where c_s and h_s are the area and energy constants of the sth problem. The motions take place along ellipses if $0<|e_s|<1$.

We may assume without loss of generality that the motions (6.3) take place in the (x_s, y_s) planes, that is, $z_s=0$ in each motion. In the frame of reference $\xi\eta\zeta$, therefore, each of the planets $\mathbf{P}_s$ has its own orbital plane (Π_s), which makes an angle φ_s with the $\xi\eta$ plane. For the family symmetric relative to the set $\mathbf{M}_1$ the semi-major axes of the ellipses coincide with the ξ axis, while those in the second family lie on the straight lines $\xi_s=0$, $\eta_s\cos\varphi_s+\zeta_s\sin\varphi_s=0$; $s=1,\dots,N$ (Fig. 1). For the first family

$$x_s=r_s^*(\theta_s)\cos\theta_s,\ \ y_s=r_s^*(\theta_s)\sin\theta_s,\ \ z_s=0\tag{6.4}$$

and for the second

$$x_s=r_s^*(\theta_s)\sin\theta_s,\ \ y_s=r_s^*(\theta_s)\cos\theta_s,\ \ z_s=0\tag{6.5}$$

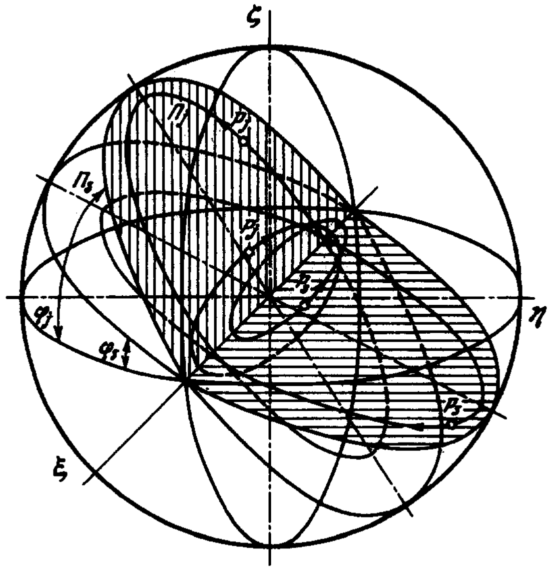

Fig. 1.

These motions are periodic in the sth subsystem. For the whole generating system, solution (6.3) forms a $2N$-parametric family, relative to the initial data (c_s, h_s), of conditionally periodic motions. Among these there are periodic motions, for which the following conditions hold

$$n_s = \frac{\sqrt{k_s}}{a_s^{3/2}} = l_s\omega \quad (l_s \in \mathbb{N}), \quad \lambda_s = a_s(1 - e_s^2) \quad (s = 1, \ldots, N) \tag{6.6}$$

where n_s are the average motions, a_s are the semi-major axes of the ellipses and ω is a certain positive number. Relations (6.6) imply that the average motions relate to one another like integers.

We thus have $N - 1$ conditions (6.6) imposed on the $2N$ constants c_s and h_s. Consequently, we have an $(N + 1)$-family, dependent on the initial conditions, of periodic motions. Now, taking into account that system (6.1) depends on $N - 1$ essential parameters $\varphi_s - \varphi_j$, whose values are also determined by the initial data, we obtain a $2N$-family of symmetric elliptical motions of the generating system. There are six such families since, when the change of variables $(\xi, \eta, \zeta) \to (\mathbf{x}, \mathbf{y}, \mathbf{z})$ is made, (ξ, η) may be replaced by any of three pairs, and for each pair there are two versions, (6.4) or (6.5). We also note that conditions (6.6) are imposed only on the semi-axes of the ellipses, not affecting the eccentricities e_s, while the direction of motion of each planet in its ellipse is determined independently. Symmetric periodic orbits obtained by continuation with respect to a parameter μ of the elliptical solutions (6.4) or (6.5) will be called periodic orbits of the third kind if not all the differences $\varphi_s - \varphi_j$ $(s \neq j)$ vanish.

To fix our ideas, let us consider solution (6.4) of the generating system, assuming that in its neighbourhood

$$\begin{aligned} &x_s + iy_s = r_s^*(\theta_s)\exp(i\theta_s)(1 + p_s), \quad x_s - iy_s = r_s^*\exp(-i\theta_s)(1 + q_s) \\ &z_s = r_s^*(\theta_s)\sigma_s \quad (s = 1, \ldots, N) \end{aligned} \tag{6.7}$$

Then we obtain

$$\begin{aligned} &\frac{d^2 p_s}{d\theta_s^2} + 2i\frac{dp_s}{d\theta_s} + \frac{1 + p_s}{1 + e_s\cos\theta_s}R_s(\mathbf{p}, \mathbf{q}, \boldsymbol{\sigma}) + \mu\frac{r_s^{*3}e^{-i\theta_s}}{c_s^2}(X_s^* + iY_s^*) = 0 \\ &\frac{d^2 q_s}{d\theta_s^2} - 2i\frac{dq_s}{d\theta_s} + \frac{1 + q_s}{1 + e_s\cos\theta_s}R_s(\mathbf{p}, \mathbf{q}, \boldsymbol{\sigma}) + \mu\frac{r_s^{*3}e^{i\theta_s}}{c_s^2}(X_s^* - iY_s^*) = 0 \\ &\frac{d^2\sigma_s}{d\theta_s^2} + \left(1 + \frac{R_s(\mathbf{p}, \mathbf{q}, \boldsymbol{\sigma})}{1 + e_s\cos\theta_s}\right)\sigma_s + \mu\frac{r_s^{*3}}{c_s^2}Z_s^* = 0 \end{aligned} \tag{6.8}$$

$$R_s(\mathbf{p},\mathbf{q},\boldsymbol{\sigma}) = [(1+p_s)(1+q_s)+\sigma_s^2]^{-3/2} - 1 \ (s=1,\ldots,N)$$

where X_s^*, Y_s^*, Z_s^*, are obtained by substituting the new variables p_s, q_s and σ_s as in formulae (6.7) into the functions X_s, Y_s and Z_s, respectively.

The independent variable in system (6.8) is the angle $\theta = \omega t$, the right-hand sides are 2π-periodic functions of θ, and $\pm\theta_s = l_s(\theta) + f_s(l_s\theta)$ [15], where f_s is a Fourier series in $l_s\theta$ which is absolutely convergent provided $|e| < \bar{e}$ (the Laplace limit), and the sign of θ_s is the same as that of c_s. Bearing this in mind, it is nevertheless convenient to retain the angle θ_s as the variable in the sth subsystem for our further discussion.

We also note the important fact that system (6.8) is reversible and invariant with respect to the change of variables $(t, \mathbf{p}, \mathbf{q}, \sigma) \to (t, \mathbf{q}, \mathbf{p}, -\sigma)$.

We need one more change of variables: $\sigma_s = \varepsilon\chi_s$ $(s = 1, \ldots, N)$, $\varepsilon^3 = \mu$. Then the equation for χ_s is

$$\frac{d^2\chi_s}{d\theta_s^2} + \left[1 + \frac{R_s}{1+e_s\cos\theta_s}\right]\chi_s + \varepsilon^2\Phi_s + \varepsilon^3\psi_s \ (s=1,\ldots,N)$$

$$\Phi_s = \frac{r_s^{*3}}{c_s^2}\sum k_j^*\left[\frac{1}{r_{sj}^3} - \frac{1}{r_j^3}\right]\frac{r_j^*\sin\theta_j}{1+e_s\cos\theta_s}\sin(\varphi_s - \varphi_j)$$

$$\Psi_s = \frac{r_s^{*3}}{c_s^2}\sum k_j^*\left[\frac{r_s^*\chi_s}{r_{sj}^3} - \left(\frac{1}{r_{sj}^3} - \frac{1}{r_j^3}\right) r_j^*\chi_j\cos(\varphi_s - \varphi_j)\right]$$

Hence, also using the first two groups of Eqs (6.8) and setting $\varepsilon = 0$, we obtain the generating system

$$\begin{aligned}
&\frac{d^2 p_s}{d\theta_s^2} + 2i\frac{dp_s}{d\theta_s} + \frac{1+p_s}{1+e_s\cos\theta_s}R_s^* = 0\\
&\frac{d^2 q_s}{d\theta_s^2} - 2i\frac{dq_s}{d\theta_s} + \frac{1+q_s}{1+e_s\cos\theta_s}R_s^* = 0\\
&\frac{d^2\chi_s}{d\theta_s^2} + \left[1 + \frac{R_s^*}{1+e_s\cos\theta_s}\right]\chi_s = 0, \ R_s^* = [(1+p_s)(1+q_s)]^{-3/2} - 1 \ (s=1,\ldots,N)
\end{aligned} \tag{6.9}$$

It is obvious that the subsystem of equations for p_s and q_s splits off the equations for χ_s. This subsystem has a unique (zero) periodic solution if a finite number of eccentricities are excluded from consideration [14]. Thus, the generating system (6.9) admits of a unique family of periodic solutions, symmetric relative to the set $\mathbf{M}_1^* = \{\mathbf{p}, \mathbf{q}, \chi, \mathbf{p}', \mathbf{q}', \chi' : \mathbf{p} = \mathbf{q}, \mathbf{p}' = \mathbf{q}', \chi = \mathbf{0}\}$

$$p_s = 0, \ q_s = 0, \ \chi_s = \alpha_s\sin\theta_s(\alpha_s = \text{const}); \ s = 1,\ldots,N \tag{6.10}$$

The dimension of the fixed set $\mathbf{M}_1^*$ is $3N$—half the dimension of the phase space. In the plane problem ($\chi_s = 0$; $s = 1, \ldots, N$) the solution $p_s = q_s = 0$ $(s = 1, \ldots, N)$ is a generating solution [14] and Ra $= 2N$; hence in system (6.9), considering solution (6.10), we also have Ra $= 2N < 3N$.

We will now determine the conditions for the continuation of solution (6.10) with respect to ε, using the theory presented in Section 4. The equations for χ_s do not contain terms linear in ε. The terms of order ε^2 have the form

$$U_{1s} = -\frac{3\chi_s^3}{2(1+e_s\cos\theta_s)} + \Phi_s \ (s=1,\ldots,N)$$

It therefore follows from (4.4) that the amplitudes $|\alpha_s|$ of the generating solution must satisfy the equations

$$\frac{3}{2}\alpha_s^3\int_0^{\pi}\frac{\sin^4\theta_s d\theta}{1+e_s\cos\theta_s} = A_s^* \tag{6.11}$$

$$A_s^* = \int_0^\pi g_s(\theta)\sin\theta_s \sin\theta_j d\theta, \quad g_s(\theta) = \frac{r_s^{*3}}{c_s^2}\sum k_j^* \left[\frac{1}{r_{sj}^{*3}} - \frac{1}{r_j^{*3}}\right]\frac{r_j^* \sin(\varphi_s - \varphi_j)}{1 + e_j \cos\theta_j}$$

$$r_{sj}^{*2} = r_s^{*2} + r_j^{*2} - 2r_s^* r_j^* [\cos\theta_s \cos\theta_j + \sin\theta_s \sin\theta_j \cos(\varphi_s - \varphi_j)] \ (s = 1, \ldots, N)$$

It is obvious that if $A_s^* \neq 0$ ($s = 1, \ldots, N$), all the roots α_s of Eq. (6.11) are simple, so that the continuability conditions of Theorem 3 are satisfied.

The integrals A_s^* depend on the $2N$ parameters e_s, $\varphi_s - \varphi_1$ ($s = 1, \ldots, N$), ω. In the space of these parameters the condition $A_j^* = 0$ defines a $2N - 1$-dimensional manifold for which (6.10) is not a generating solution. For fixed eccentricities e_s and frequency ω the condition $A_j^* = 0$ defines the planes for which the problem of the existence of three-dimensional periodic orbits requires further investigation. In any case, almost all three-dimensional orbits (6.4) and (6.10) are generating motions.

Analogous calculations from the family (6.5) lead to the problem of continuation with respect to the parameter ε of periodic solutions $p_s = 0$, $q_s = 0$, $\chi_s = \beta_s \cos\theta_s$ (β_s = const), $s = 1, \ldots, N$, which are symmetric relative to the fixed set $\mathbf{M}_2^* = \{\mathbf{p}, \mathbf{q}, \chi, \mathbf{p}^{\cdot}, \mathbf{q}^{\cdot}, \chi^{\cdot}: \mathbf{p} = \mathbf{q}, \mathbf{p}^{\cdot} = \mathbf{q}^{\cdot}, \chi^{\cdot} = \mathbf{0}\}$.

In these circumstances the amplitudes $|\beta_s|$ of the generating equation satisfy the equations

$$\frac{3}{2}\beta_s^3 \int_0^\pi \frac{\cos^4\theta_s d\theta}{1 + e_s \cos\theta_s} = B_s^*, \quad B_s^* = \int_0^\pi g_s(\theta)\cos\theta_s \cos\theta_j d\theta$$

We can now determine the form of the three-dimensional orbits more precisely. In the first approximation with respect to ε we obtain

$$x_s = r_s^*(\theta_s)\cos\theta_s + O(\varepsilon^2), \quad y_s = r_s^*(\theta_s)\sin\theta_s + O(\varepsilon^2), \quad z_s = \varepsilon\alpha_s r_s^*(\theta_s)\sin\theta_s + O(\varepsilon^2) \tag{6.12}$$

$$x_s = r_s^*(\theta_s)\sin\theta_s + O(\varepsilon^2), \quad y_s = r_s^*(\theta_s)\cos\theta_s + O(\varepsilon^2), \quad z_s = \varepsilon\beta_s r_s^*(\theta_s)\cos\theta_s + O(\varepsilon^2) \tag{6.13}$$

for cases (6.4) and (6.5), respectively. Hence it follows that when $\mu \neq 0$ the symmetric periodic orbits lie in an $O(\mu^{1/3})$-neighbourhood of the planes Π_s. These motions are $2\pi/\omega$-periodic with respect to the time t.

Theorem 5. For sufficiently small $\mu = \max_s\{M_s/M_0\}$, the N-planet problem of a main body with mass M_0 and planets $\mathbf{P}_1, \ldots, \mathbf{P}_N$ with masses $M_1, \ldots, M_N$, respectively, admits of two families of symmetric, nearly elliptical, periodic orbits for all values of the eccentricities $e_s(0 < |e_s| < 1)$, with the possible exception of a finite number of critical values. The orbit of a planet $\mathbf{P}_s$ lies in an $O(\mu^{1/3})$-neighbourhood

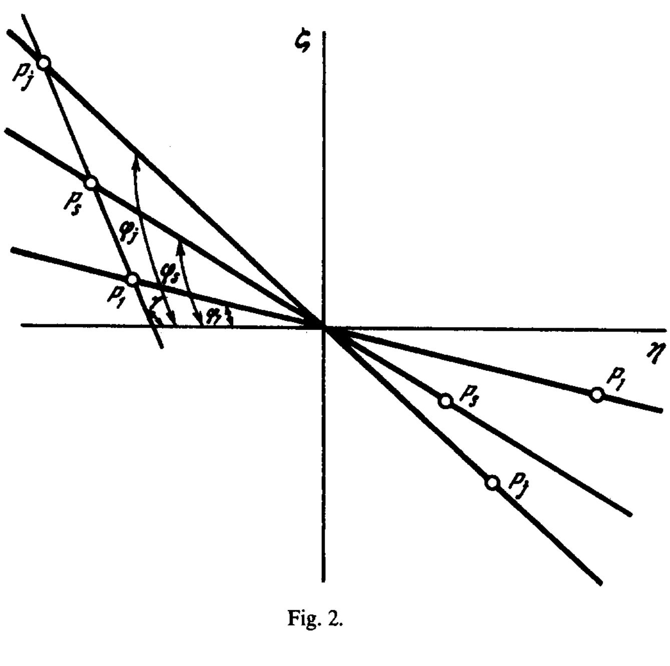

Fig. 2.

of the planes Π_s and is described by formulae (6.12) or (6.13), and all the planes pass through the same fixed straight line—the line of nodes; the angle $\varphi_s - \varphi_j$ between the planes Π_s and Π_j is arbitrary. The average motions along the orbits equal the average motions along the ellipses (6.4) or (6.5) and relate to one another as natural numbers; the motion of the whole system is periodic. At times a multiple of a half-period all the planets in the first family lie along the line of nodes (a parade of planets); in the second family, they intersect the same fixed plane, which contains the apsidal curves in the unperturbed problem ($\mu = 0$).

It is clear that in the second family of orbits as well the planets $\mathbf{P}_1, \ldots, \mathbf{P}_N$ may form a straight line at times that are multiples of the half-period (Fig. 2). Up to terms of the order of $\mu^{1/3}$ the distance $\mathbf{SP}_s$ at these times is equal to $a_s(1 + e_s)$, while in the case when $e_s > 0$ the planet $\mathbf{P}_s$ in a generating solution traverses the apocentre, otherwise—the pericentre. If the straight line $\mathbf{L}$ on which the planets are situated makes an angle γ with the η axis, then

$$\frac{a_s(1+e_s)}{a_1(1+e_1)} = \frac{\sin(\gamma+\varphi_1)}{\sin(\gamma+\varphi_s)}$$

In the course of one half-period the planets will generally not line up again in a straight line. This is illustrated by Fig. 2, in which $e_1 < 0$, $e_s > 0$, $e_j > 0$. Consequently, for the second family of orbits, a parade of planets is observed at times that are multiples of the period $2\pi/\omega$. Only in the case $e_s = e_1$ ($s = 2, \ldots, N$) is a parade of planets observed twice in one period.

I wish to thank V. V. Rumyantsev, who drew my attention to the phenomenon of resonance in the Solar system.

This research was supported financially by the Russian Foundation for Basic Research (97-01-00538).

REFERENCES

1. TKHAI, V. N., Fixed sets and symmetric periodic motions of reversible mechanical systems. *Prikl. Mat. Mekh.*, 1996, **60**, 6, 959–971.
2. TKHAI, V. N., Non-linear oscillations of reversible systems. *Prikl. Mat. Mekh.*, 1995, **59**, 1, 38–50.
3. TKHAI, V. N., Symmetric periodic orbits of the third kind in the *N*-planet problem. Resonance and parade of planets. *Dokl. Ross. Akad. Nauk*, 1996, **350**, 1, 52–55.
4. MALKIN, I. G., *Some Problems of the Theory of Non-linear Oscillations*. Gostekhizdat, Moscow, 1956.
5. LYAPUNOV, A. M., On series proposed by Hill to represent motions of the Moon. In: *Collected Papers. Izd. Akad. Nauk SSSR*, Moscow, 1954, **1**, 418–476.
6. TKHAI, V. N., Symmetric periodic orbits in the restricted three-body problem. *Kosmich. Issled.*, 1997, **35**, 2, 164–171.
7. HILL, G. W., Researches in the lunar theory. *Am. J. Math.*, 1878, 1, 5–26, 129–147, 245–260.
8. PROSKURYAKOV, A. P., *Poincaré's Method in the Theory of Non-linear Oscillations*. Nauka, Moscow, 1977.
9. KHOLOSTOVA, O. V., On the motion of a nearly Hamiltonian system with one degree of freedom in the case of resonance in the forced oscillations. *Prikl. Mat. Mekh.*, 1996, **60**, 3, 405–412.
10. MATVEYEV, M. V. and TKHAI, V. N., Stability of periodic reversible systems. *Prikl. Mat. Mekh.*, 1993, **57**, 1, 3–11.
11. BRYUNO, A. D., The analytical form of ordinary differential equations. *Trudy Mosk. Mat. Obshch.*, 1971, **25**, 119–262; 26, 199–239.
12. DEVANEY, R. L., Reversible diffeomorphisms and flows. *Trans. Am. Math. Soc.*, 1995, **218**, 89–113.
13. MOULTON, F. R., BUCHMAN, D. and BUCH, T., *Periodic Orbits*. Carnegie Institute, Washington, 1920.
14. TKHAI, V. N., Symmetric periodic orbits of the many-body problem. Resonance and parades of planets. *Prikl. Mat. Mekh.*, 1995, **59**, 3, 355–365.
15. DUBOSHIN, G. N., *Celestial Mechanics. Fundamental Problems and Methods*. Nauka, Moscow, 1968.

Translated by D.L.